Oneindigheid / Infinity

Benedikt Löwe

Logica en de Linguistic Turn
18 October 2017, 16:15-18:00

Achilles & the Tortoise

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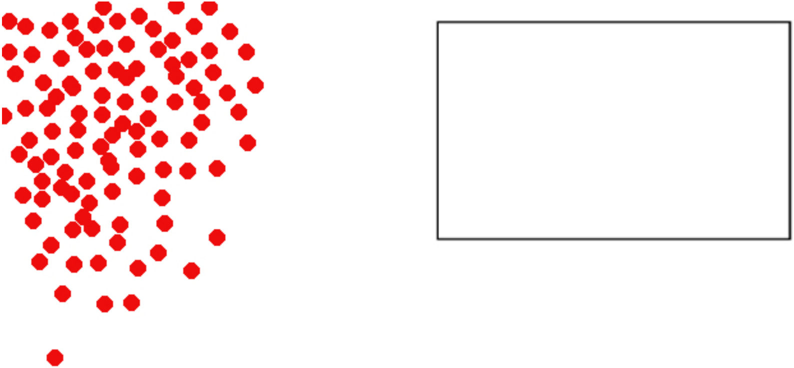
Zeno of Elea (5th century BC)

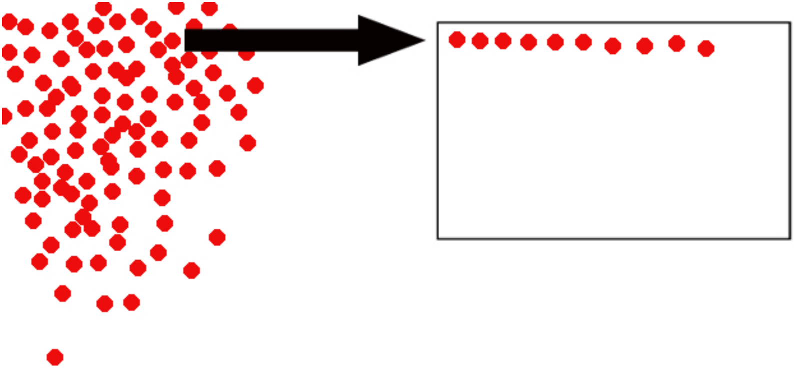
[T]he argument says that it is impossible for [Achilles] to overtake the tortoise when pursuing it. For in fact it is necessary that what is to overtake [something], before overtaking [it], first reach the limit from which what is fleeing set forth. In [the time in] which what is pursuing arrives at this, what is fleeing will advance a certain interval, even if it is less than that which what is pursuing advanced ... And thus in every time in which what is pursuing will traverse the [interval] which what is fleeing, being slower, has already advanced, what is fleeing will also advance some amount. (Simplicius, On Aristotle's Physics, 1014.10)

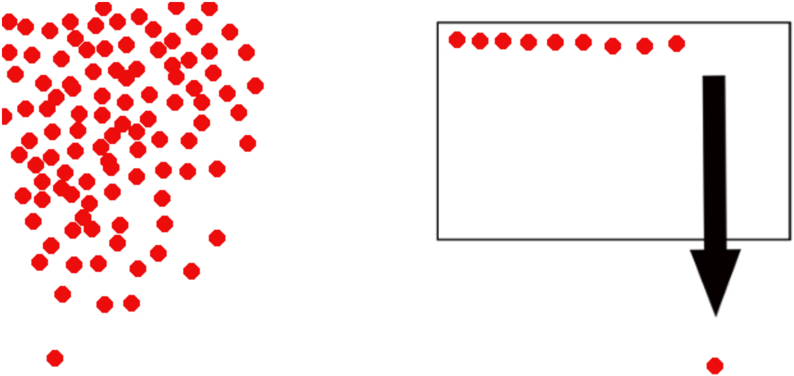
Thomson's Lamp

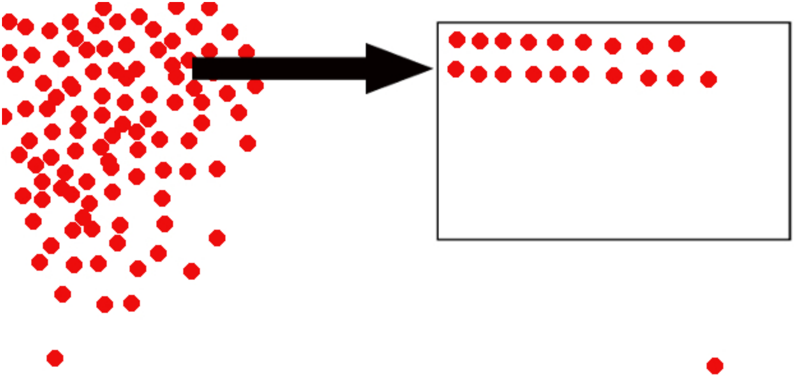
James F. Thomson, Tasks and Super-Tasks, *Analysis* 15:1, 1954, 1–13.

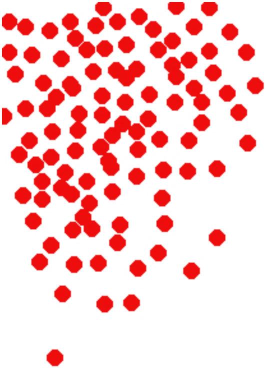
There are certain reading-lamps that have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and you press the button the lamp goes off. So if the lamp was originally off, and you pressed the button an odd number of times, the lamp is on, and if you pressed the button an even number of times the lamp is off. Suppose now that the lamp is off, and I succeed in pressing the button an infinite number of times, perhaps making one jab in one minute, another jab in the next half-minute, and so on. ... After I have completed the whole infinite sequence of jabs, i.e., at the end of the two minutes, is the lamp on or off? It seems impossible to answer this question. It cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction.

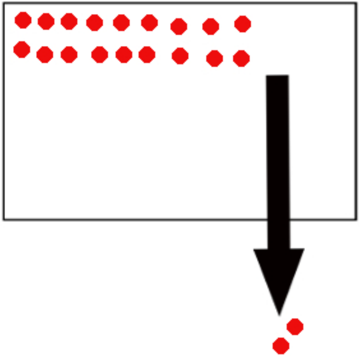


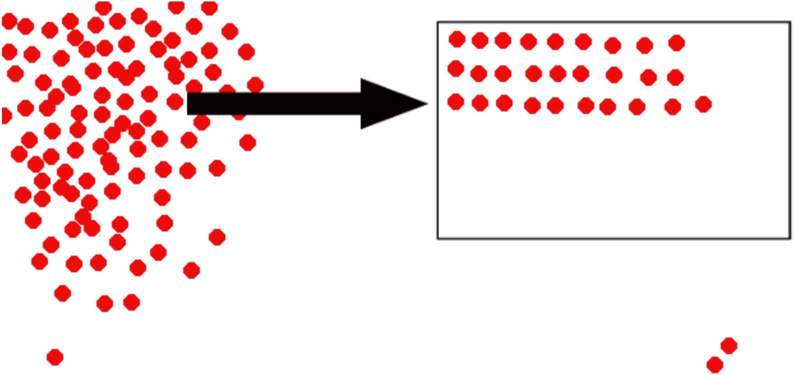


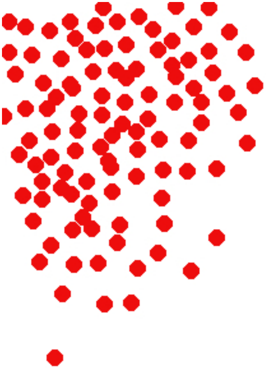


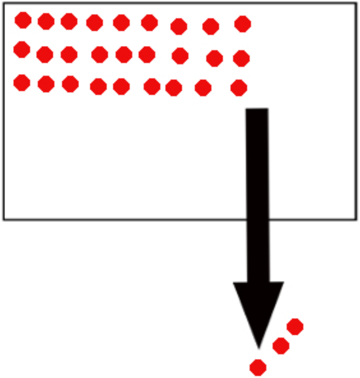


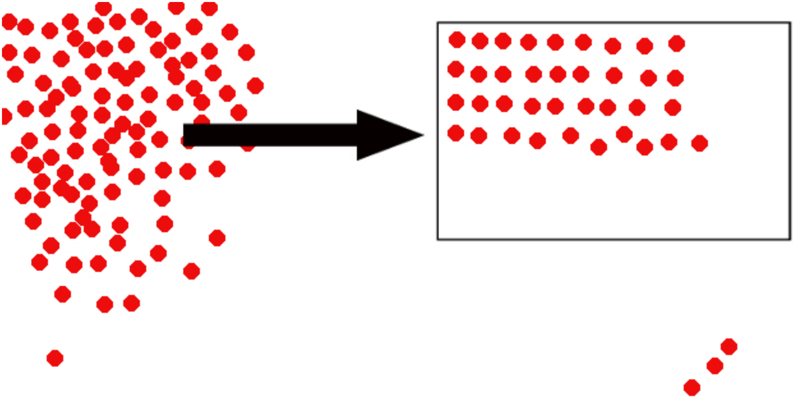


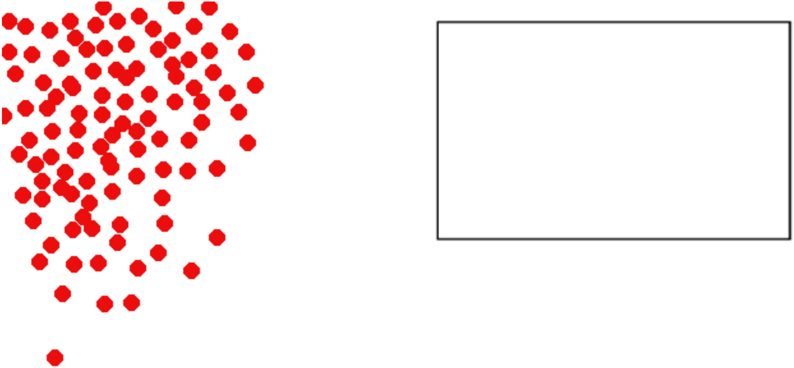


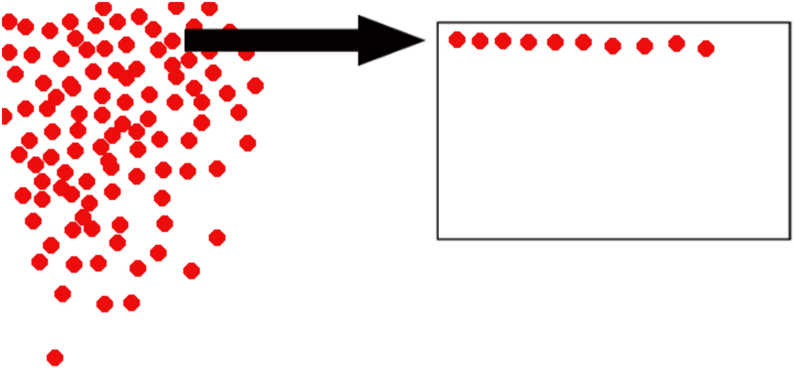


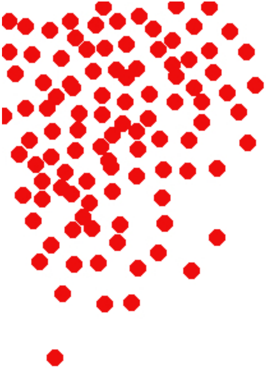


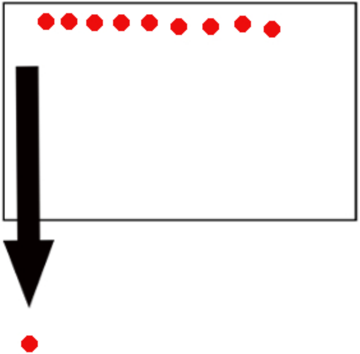


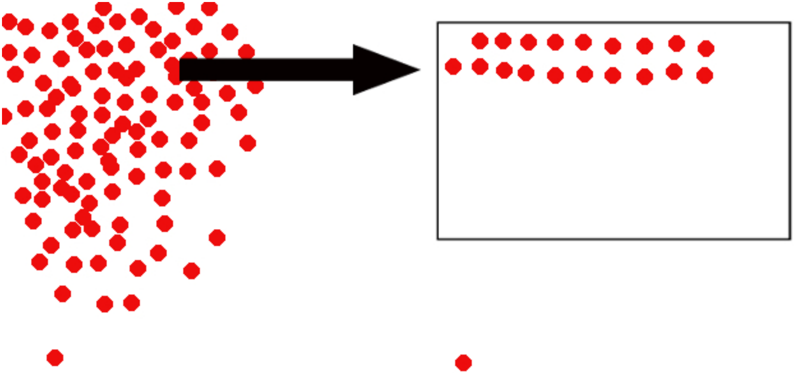


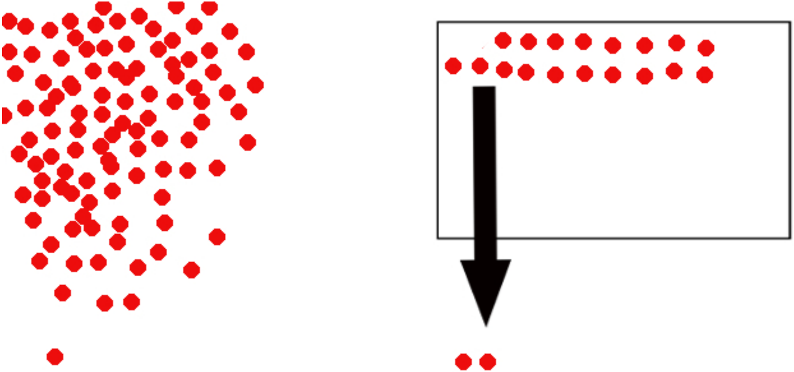


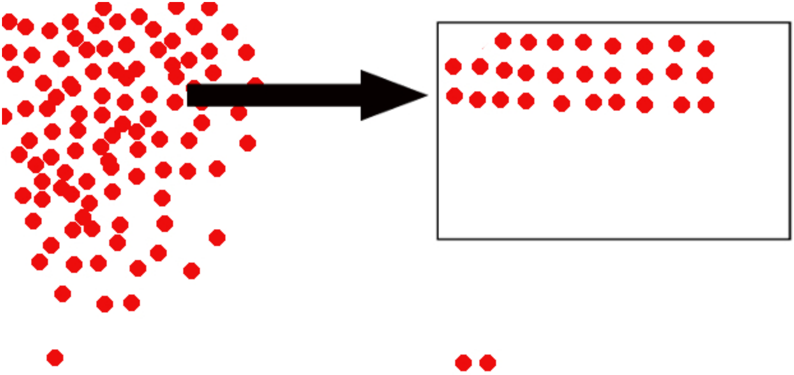


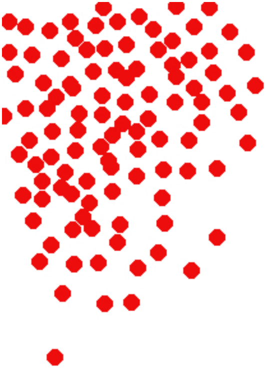


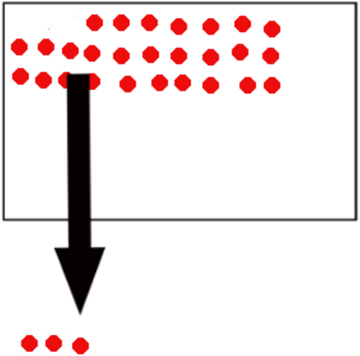


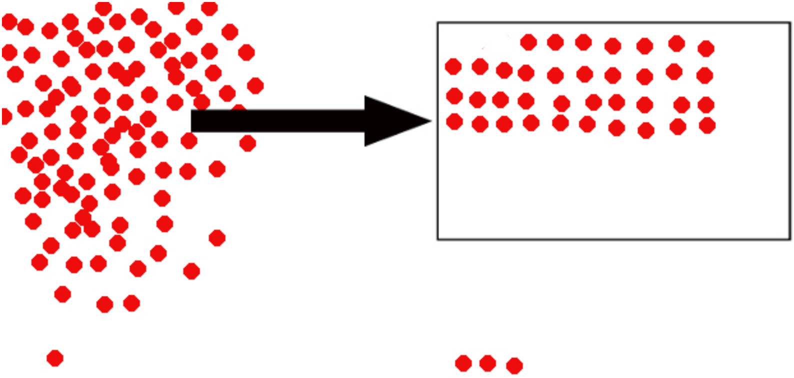












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- *Epistemology.*
  The scientific method is *inductive*; mathematics is *deductive*. Mathematics as template for purely rational thought (*Ethica, ordine geometrico demonstrata*). Certainty about knowledge.

Aristotle

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Pieter Sjord Hasper, Aristotle on Infinity, Kolloquium-Vorlesung, Berlin, 2008.

Bernhard Bolzano (1781-1848)

Paradoxien des Unendlichen (1851)

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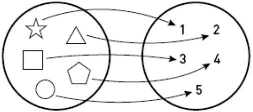

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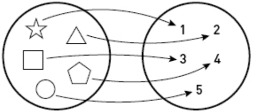

There are as many objects on the left hand side as there are numbers on the right hand side, since I can match them by a one-to-one correspondence.

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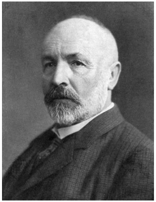
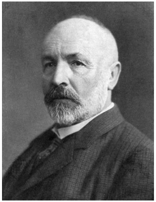
$$\begin{tikzcd}
0 \arrow[d] & 1 \arrow[dr] & 2 \arrow[drr] & 3 \arrow[drrr] & 4 \arrow[drrrr] & 5 \arrow[drrrrr] & 6 \arrow[drrrrrr] & 7 \arrow[drrrrrrr] & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \cdots \\
\mathbf{0} & \underline{1} & \mathbf{2} & \underline{3} & \mathbf{4} & \underline{5} & \mathbf{6} & \underline{7} & \mathbf{8} & \underline{9} & \mathbf{10} & \underline{11} & \mathbf{12} & \underline{13} & \mathbf{14} & \cdots
\end{tikzcd}$$



Georg Cantor (1845-1918)

Measuring the size of the infinite



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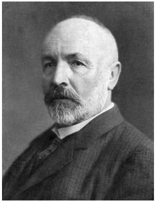

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Cantor realized that the definitions are asymmetric: to be *of the same size*, it is enough to have **some** one-to-one correspondence between X and Y, but for X to have *more elements* that Y, it is enough that **one particular** one-to-one correspondence does not exhaust X.

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The box & the red balls. By removing a ball to the rubbish heap, we create a one-to-one correspondence between the infinite number of rounds and the set of red balls on the rubbish heap. The one-to-one correspondence we get when we remove the newest ball is different from the one we get when we remove the oldest ball. The different answers to the question about the size of the remaining balls is related to the properties of the created one-to-one correspondences.

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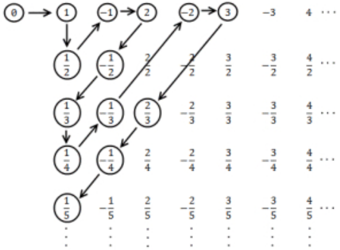

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Now suppose infinitely many new guests arrive, sweaty and short-tempered. No problem. The unflappable manager moves the occupant of room 1 to room 2, room 2 to room 4, room 3 to room 6, and so on. This doubling trick opens up all the odd-numbered rooms—infinitely many of them—for the new guests.

Sets of different infinite sizes (1).

Since $\mathbb{N}$ and $\mathbb{Q}$ have the same size, maybe, there are no different infinite sizes? I.e., if X and Y are infinite, then there is a one-to-one correspondence between them?

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The answer to this question is negative:

Consider the set of infinite 0-1-sequences, e.g.,

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and call it $\mathcal{C}$.

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This set is clearly infinite, but we shall see that this set cannot be in one-to-one correspondence with the natural numbers.

Sets of different infinite sizes (2).

Theorem (Cantor). The set of natural numbers $\mathbb{N}$ is smaller than the set of infinite 0-1-sequences $\mathcal{C}$.

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We shall show that there must be some sequence d that is not in correspondence with any natural number. Define the ith digit of d by looking up the ith digit of s_i and flipping it, i.e., $d(i) := 1 - s_i(i)$.

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\vdots & & & & & & & & &
\end{array}$$

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$$d = 100110...$$

Sets of different infinite sizes (2).

Theorem (Cantor). The set of natural numbers $\mathbb{N}$ is smaller than the set of infinite 0-1-sequences $\mathcal{C}$.

Proof. Suppose that there is a function that assigns to each natural number n a 0-1-sequence s_n.

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The Continuum Problem

David Hilbert (1862-1943)

Hilbert's Problems

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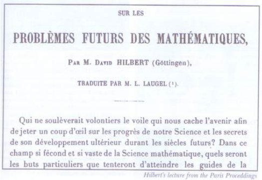

SUR LES

PROBLÈMES FUTURS DES MATHÉMATIQUES,

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TRADUITE PAR M. L. LAUGEL (1).

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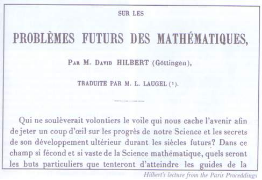

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The **Continuum Problem** was number one on this list.

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Emil du Bois-Reymond
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Emil du Bois-Reymond, Über die Grenzen des Naturerkennens, 1872

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Emil du Bois-Reymond, Über die Grenzen des Naturerkennens, 1872

"Wir dürfen nicht denen glauben, die heute mit philosophischer Miene und überlegenem Tone den Kulturuntergang prophezeien und sich in dem Ignorabimus gefallen. Für uns gibt es kein Ignorabimus, und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht. Statt des törichten Ignorabimus heisse im Gegenteil unsere Losung: Wir müssen wissen, Wir werden wissen."

(D. Hilbert, Radio Address, 8 September 1930)

Ignoramus et ignorabimus (2).

The Continuum Problem. Are there sets $A \subseteq \mathbb{R}$ that are strictly bigger than the set of natural numbers and strictly smaller than the set of real numbers?

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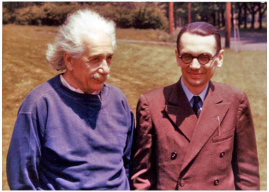

Kurt Gödel (1906-1978)

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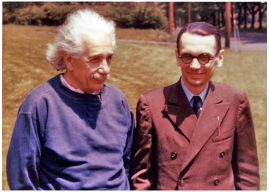

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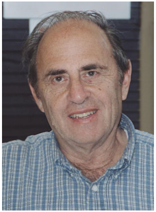

Paul Cohen (1934-2007)

Theorem (1962). It is not possible to refute the existence of such a set A.

Ignorabimus?

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Possible answers:

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- There is a fundamental problem with infinity.

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Possible answers:

- There is a fundamental problem with infinity.
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- Du Bois-Raymond was right, *ignorabimus*: there are fundamental limits to the deductive method.
- Mathematical logic should study those statements that are neither provable nor refutable.